ELEN E3401: Electromagnetics

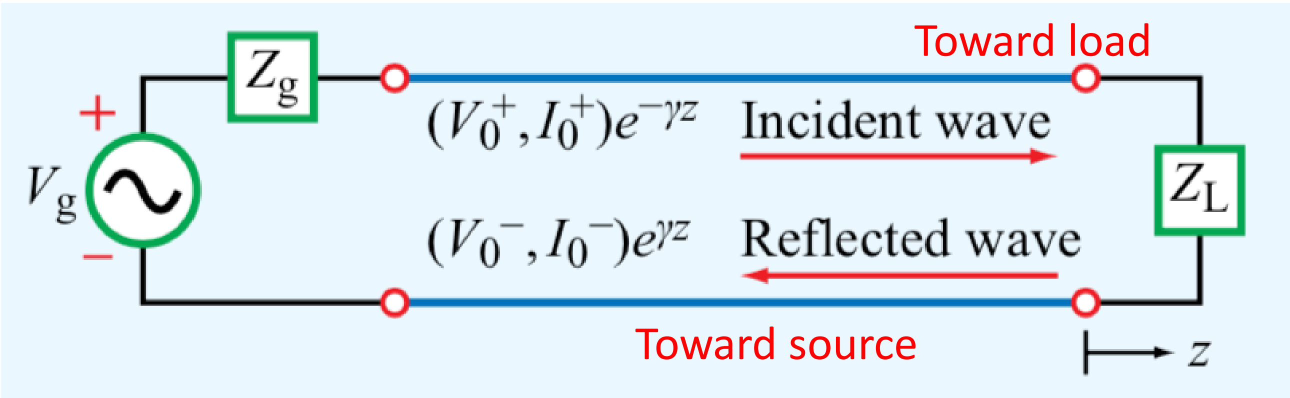
Spring 2025

Prof. Keren Bergman

Lecture #5

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Traveling wave solutions



4 unknowns:
(V_0^+, I_0^+) and (V_0^-, I_0^-)

Wave amplitudes of incident/reflected

Relate the 4 wave amplitudes: I_0^+ and I_0^- to V_0^+ and V_0^-

$$\widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$ Total voltage

$$\widetilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$ Total current

$$-\frac{d\widetilde{V}(z)}{dz} = (R' + j\omega L')\widetilde{I}(z)$$ ← Transmission line voltage phasor equation

$$-\frac{d\widetilde{V}(z)}{dz} = \gamma V_0^+ e^{-\gamma z} - \gamma V_0^- e^{\gamma z} = (R' + j\omega L')\widetilde{I}(z)$$

$$\Rightarrow \quad \widetilde{I}(z) = \frac{\gamma}{R' + j\omega L'}[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}].$$

Characteristic Impedance

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \qquad \tilde{I}(z) = \frac{\gamma}{R' + j\omega L'}\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right]$$

$$I_0^+ = \frac{V_0^+ \gamma}{(\mathrm{R}' + j\omega L')} \quad and \quad I_0^- = \frac{-V_0^- \gamma}{(\mathrm{R}' + j\omega L')}$$

Define the ratio: $\frac{V_0^+}{I_0^+} = Z_0 = \frac{-V_0^-}{I_0^-}$

- Z_0 is the characteristic impedance
- Z_0 is the ratio of <u>voltage amplitude</u> to <u>current amplitude</u> for each individual traveling wave
- *Not* ratio of total $\tilde{V}(z)$ and $\tilde{I}(z)$

Characteristic impedance

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad \Omega$$

Wave Propagation in Transmission Line

Voltage ~ related to $\vec{E}$, current ~ related to $\vec{H}$

$$\frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

Z_0 : characteristic impedance → ω and R', L', G', C'

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

2 equations, 2 unknowns

Wave Propagation in Transmission Line

$$V_0{}^+ = |V_0{}^+|e^{j\varphi+}$$

$$V_0{}^- = |V_0{}^-|e^{j\varphi-}$$

We will apply boundary conditions at the source and load to obtain values for $V_0{}^+$and $V_0{}^-$

Now can convert back to time domain:

$$v(z,t) = Re\left\{[\tilde{V}(z)]e^{j\omega t}\right\}$$

$$= Re\left[(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z})e^{j\omega t}\right]$$

$$= Re\left[|V_0^+|e^{j\varphi^+} e^{j\omega t} e^{-(\alpha+j\beta)z} + |V_0^-|e^{j\varphi^-} e^{j\omega t} e^{(\alpha+j\beta)z}\right]$$

$$v(z,t) = |V_0^+|e^{-\alpha z}\cos(\omega t - \beta z + \varphi^+) \longleftarrow \quad \text{+z direction}$$
$$+ |V_0^-|e^{\alpha z}\cos(\omega t + \beta z + \varphi^-) \longleftarrow \quad \text{-z direction}$$

Wave Propagation in Transmission Line

Both +z and –z propagating waves have same phase velocity:

$$u_p = f\lambda = \frac{\omega}{\beta}\ .$$

Guided wavelength

$e^{-\alpha z}$ → attenuation for +z

$e^{\alpha z}$ → attenuation for -z

Standing Wave:

- Lossless TL: α = 0
- Lossy TL: $\alpha \neq 0$

TL Wave Propagation - Summary

We obtained:

$$\begin{cases} \tilde{V}(z) = V_0{}^{+} e^{-\gamma z} + V_0{}^{-} e^{\gamma z} \\ \tilde{I}(z) = \dfrac{V_0{}^{+}}{Z_0} e^{-\gamma z} - \dfrac{V_0{}^{-}}{Z_0} e^{\gamma z} \quad \textit{where } Z_0 = \sqrt{\dfrac{R' + j\omega L'}{G' + j\omega C'}} \end{cases}$$

In general:

$$\begin{cases} V_0{}^{+} = |V_0{}^{+}| e^{j\varphi +} \\ V_0{}^{-} = |V_0{}^{-}| e^{j\varphi -} \end{cases}$$

And the time domain, instantaneous solution:

$$v(z,t) = |V_0{}^{+}| e^{-\alpha z} \underbrace{\cos(\omega t - \beta z + \varphi^{+})}_{+z \text{ wave}} + |V_0{}^{-}| e^{\alpha z} \underbrace{\cos(\omega t + \beta z + \varphi^{-})}_{-z \text{ wave}}$$

Standing wave

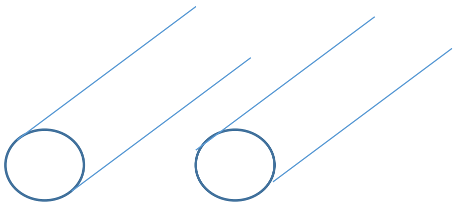
$$u_p = f\lambda = \frac{\omega}{\beta}$$

$$e^{-\alpha z} \rightarrow attenuation$$

Lossless TL: $\alpha = 0$

Lossless Transmission Line

Example: air line



Air separates 2 conductors → G'=0 because σ = 0
Also, assume high conductivity, and R'=0

Consider air line:

$\mathbf{Z_0 = 50\Omega}$

Phase constant = **20 rad/m** at **f = 700 MHz**

Q: Find L' and C' of the transmission line.

Lossless Transmission Line

Q: Find L' and C' of the air transmission line (G'=0, R'=0)

- $Z_0 = 50\Omega$
- $\beta = 20$ rad/m
- f = 700 MHz

Solution:

$$\beta = Im(\gamma) = Im\left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) = Im(j\omega\sqrt{L'C'}) = \omega\sqrt{L'C'}$$

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{j\omega L'}{jwC'}} = \sqrt{\frac{L'}{C'}}$$

$$\omega C' = \frac{\beta}{Z_0}$$

$$C' = \frac{\beta}{\omega Z_0} = 90.9\ pF/m$$

$$Z_0 = \sqrt{\frac{L'}{C'}} \rightarrow L' = 227\ nH/m$$

TL Wave Propagation - Summary

We obtained:

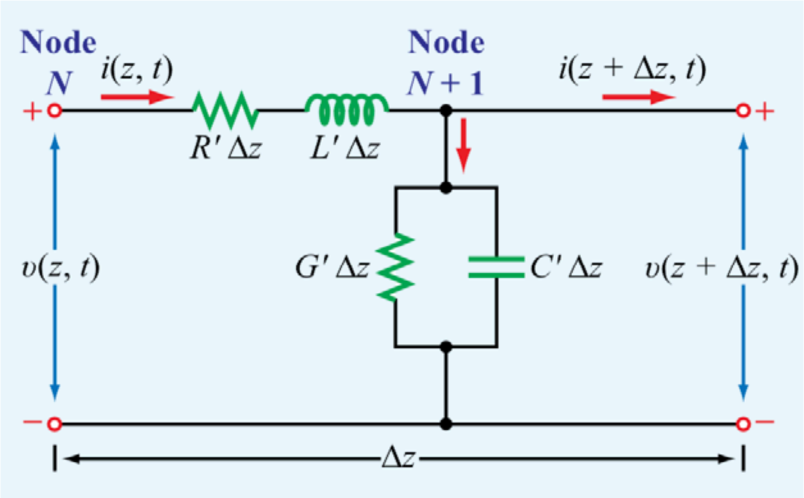
$$\tilde{V}(z) = V_0{}^{+} e^{-\gamma z} + V_0{}^{-} e^{\gamma z}$$

$$\tilde{I}(z) = \frac{V_0{}^{+}}{Z_0} e^{-\gamma z} - \frac{V_0{}^{-}}{Z_0} e^{\gamma z}$$

In general:

$$V_0{}^{+} = |V_0{}^{+}| e^{j\varphi +}$$

$$V_0{}^{-} = |V_0{}^{-}| e^{j\varphi -}$$



$$where\ Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

And the time domain, instantaneous solution:

$$v(z,t) = |V_0{}^{+}| e^{-\alpha z} \underbrace{\cos(\omega t - \beta z + \varphi^{+})}_{+\text{z wave}} + |V_0{}^{-}| e^{\alpha z} \underbrace{\cos(\omega t + \beta z + \varphi^{-})}_{-\text{z wave}}$$

Lossless transmission line

Consider a lossless line, $\gamma = j\beta$

$$\tilde{V}(z) = V_0^{\,+} e^{-j\beta z} + V_0^{\,-} e^{j\beta z}$$

$$\tilde{I}(z) = I_0^{\,+} e^{-j\beta z} + I_0^{\,-} e^{j\beta z}$$

$$\tilde{I}(z) = \frac{V_0^{\,+}}{Z_0} e^{-j\beta z} - \frac{V_0^{\,-}}{Z_0} e^{j\beta z}$$

Recall these were solutions to wave equations:

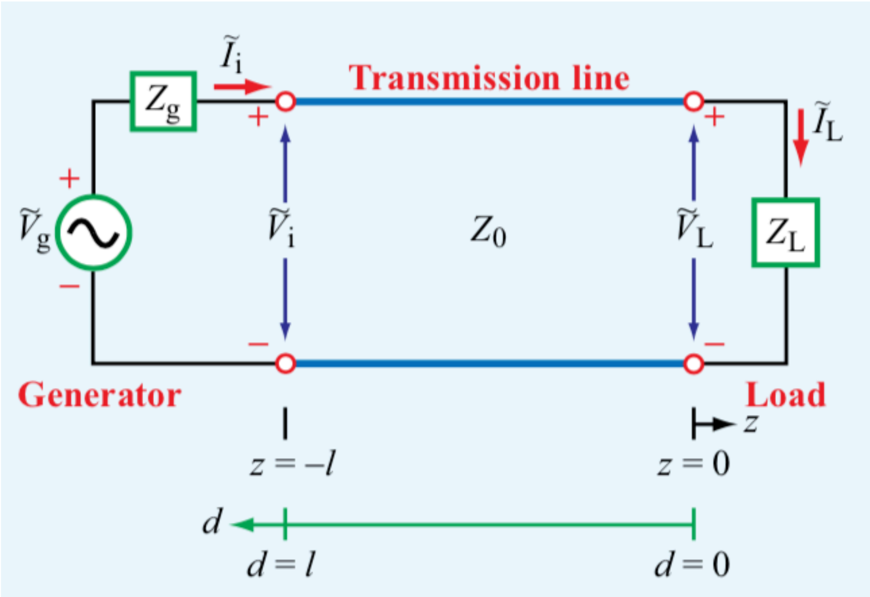
$$\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0$$

$$\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0$$

$V_0^{\,+} e^{-j\beta z}$ → incident wave travels +z (source → load)
$V_0^{\,-} e^{j\beta z}$ → reflected wave travels –z (load → source)

Full circuit with transmission line

Consider full circuit with TL, generator and load impedance Z_L (z = 0 at load)



"*z*" points from generator to load

"*d*" is distance from load $d = -z$

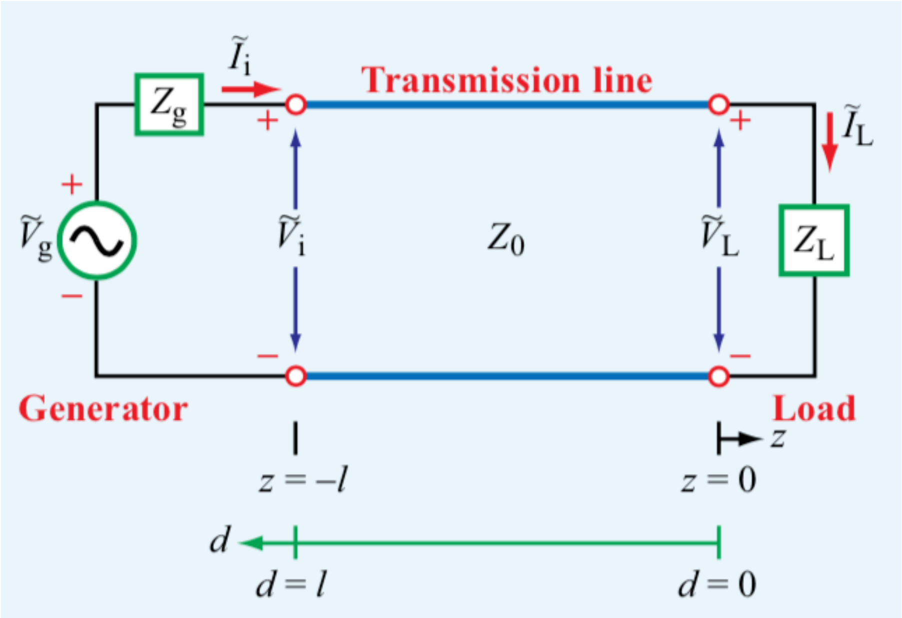
Generator at $z = -l$, $d = l$, $d = -z$

Load at $z = 0$, $d = 0$

At sending end, $z = -l$

connected to sinusoidal voltage source: $\tilde{V}_g$ and internal impedance Z_g

Voltage Reflection Coefficient



"z" points from generator to load

"d" is distance from load d = -z

At the load, $\tilde{V}_L$ = phasor voltage across load
$\tilde{I}_L$ = phasor current through load

$$load\ impedance = Z_L = \frac{\tilde{V}_L}{\tilde{I}_L}$$

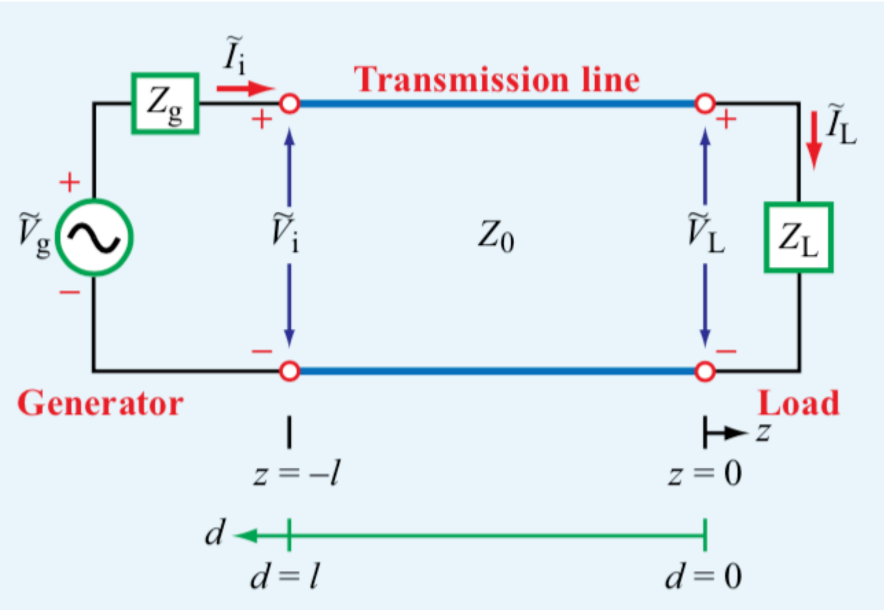
$\tilde{V}_L$, $\tilde{I}_L$ are total voltage, current at $z = 0$ (at load)

Voltage Reflection Coefficient

$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$\tilde{I}(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$



At the load, z = 0:

$$\tilde{V}_L = \tilde{V}(z=0) = V_0^+ + V_0^-$$

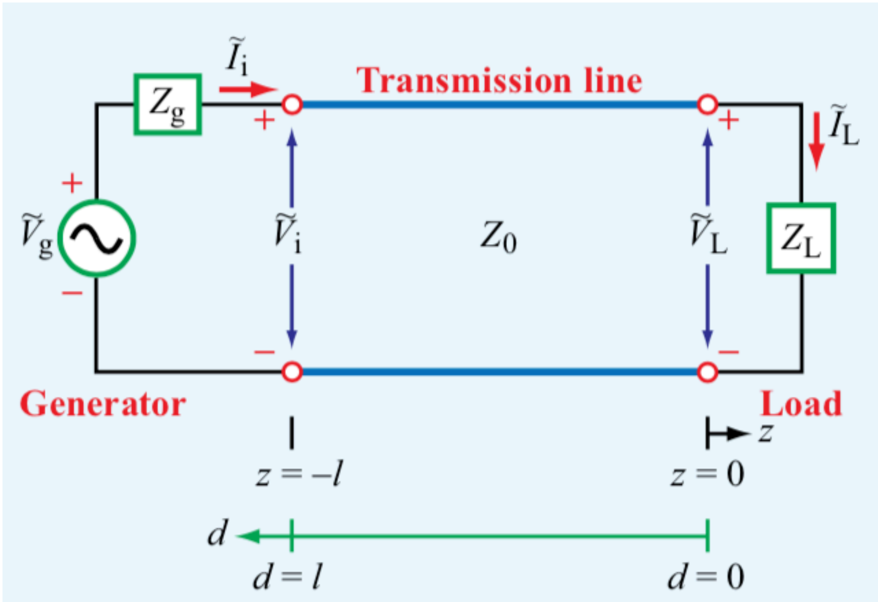
$$\tilde{I}_L = \tilde{I}(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

$$Z_L = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0$$ Load impedance

$$V_0^- = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_0^+$$

Solve for V_0^+, V_0^-

Voltage Reflection Coefficient



At the load, z = 0:

$$\widetilde{V_L} = \tilde{V}(z=0) = V_0^{\ +} + V_0^{\ -}$$

$$\widetilde{I_L} = \tilde{I}(z=0) = \frac{V_0^{\ +}}{Z_0} - \frac{V_0^{\ -}}{Z_0}$$

$$Z_L = \left(\frac{V_0^{\ +} + V_0^{\ -}}{V_0^{\ +} - V_0^{\ -}}\right) Z_0$$ Load impedance

$$V_0^{\ -} = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_0^{\ +}$$

Define the reflection coefficient for the voltage:

$$\Gamma = \frac{V_0^{\ -}}{V_0^{\ +}} \quad \left(\frac{ref}{inc}\right) \quad = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{z_L - 1}{z_L + 1} \qquad \left(z_L = \frac{Z_L}{Z_0}\right)$$

small z_L = normalized load impedance
Load impedance normalized to characteristic impedance

Voltage Reflection Coefficient

$$\frac{V_0^{\,-}}{V_0^{\,+}} = \Gamma \qquad \frac{I_0^{\,-}}{I_0^{\,+}} = -\Gamma \qquad z_L = \frac{Z_L}{Z_0}$$

Z_0 = <u>real</u> for lossless but $Z_L = R + j\omega L$ in general is complex

example

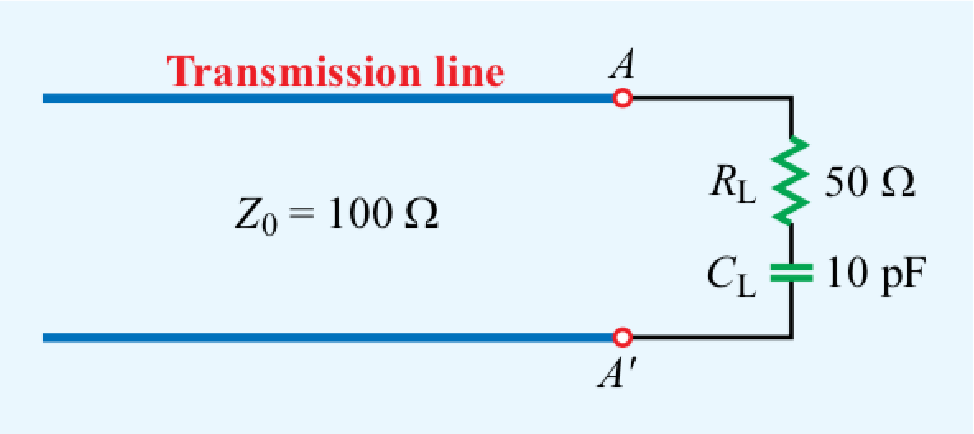
$\Gamma = |\Gamma| e^{j\theta r}$ ← Γ complex (|Γ|<=1)

1. Load is <u>matched</u> when $Z_L = Z_0$, then $\Gamma = 0$ and $V_0^{\,-} = 0$ (<u>no</u> reflection by load)
2. Load is <u>open circuit</u> when $Z_L = \infty$, then $\Gamma = 1$ and $V_0^{\,-} = V_0^{\,+}$ (reflection <u>in phase</u>, $\theta r = 0$)
3. Load is <u>short circuit</u> when $Z_L = 0$, then $\Gamma = -1$ and $V_0^{\,-} = -V_0^{\,+}$ (reflection <u>out of phase</u> $\theta r = \pm 180°$)

Example – Reflection of series RC load

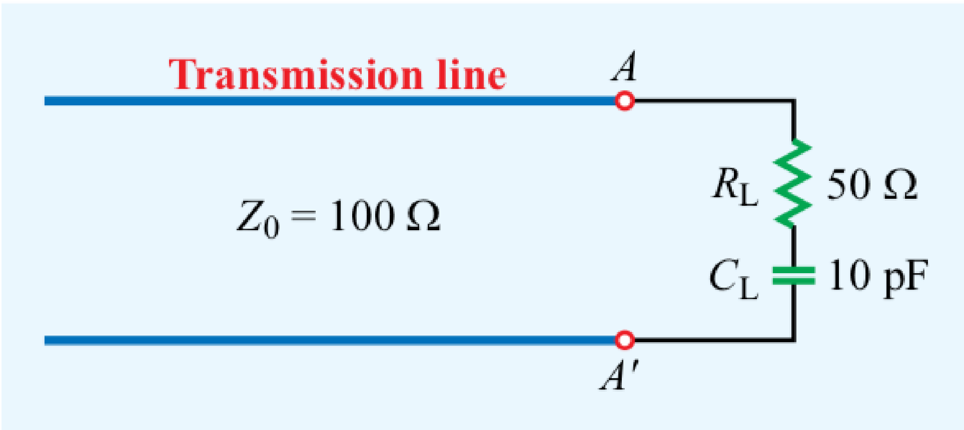
Consider a 100Ω transmission line, $Z_0 = 100\Omega$
Connected to load: resistor 50Ω in series with 10 pF capacitor:
$R_L = 50\Omega$,
$C_L = 10\text{pF} = 10^{-11}$ F

<u>Q: Find reflection coefficient for 100 MHz signal</u>



$f = 100$ MHz $= 10^8$ Hz

Example – Reflection of series RC load



f = 100 MHz = 10^8 Hz

Solution: need to find Γ

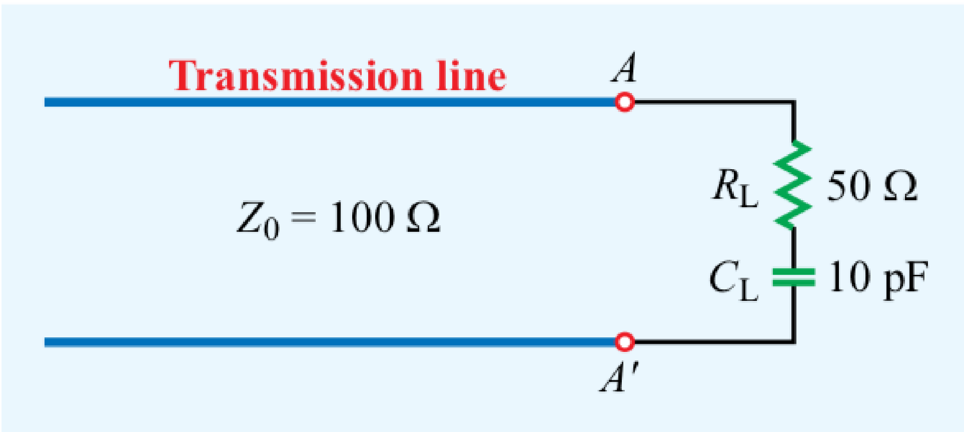
$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \Gamma = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{z_L - 1}{z_L + 1}$$

Find normalized impedance: $z_L = \frac{Z_L}{Z_0}$

$$z_L = \frac{Z_L}{Z_0} = \frac{R_L - j/(\omega C_L)}{Z_0} \qquad (Note\text{: } R + \frac{1}{j\omega C} \rightarrow R - \frac{j}{\omega C})$$

$$= \frac{1}{100}\left(50 - j\frac{1}{2\pi \times 10^8 \times 10^{-11}}\right) = (0.5 - j1.59)\Omega$$

Example – Reflection of series RC load



f = 100 MHz = 10^8 Hz

Solution:
Find the reflection coefficient (for voltage)

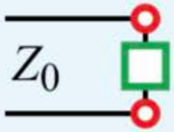
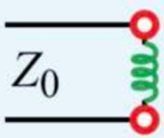
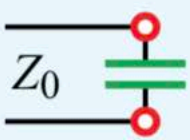
$$\Gamma = \frac{z_L - 1}{z_L + 1} = \frac{0.5 - j1.59 - 1}{0.5 - j1.59 + 1} = \frac{-0.5 - j1.59}{1.5 - j1.59} = \frac{-1.67e^{j72.6^o}}{2.19e^{-j46.7^o}}$$

$$\Gamma = -0.76\ e^{j119.3^o} \qquad (Note: -1 = e^{-j180^o})$$

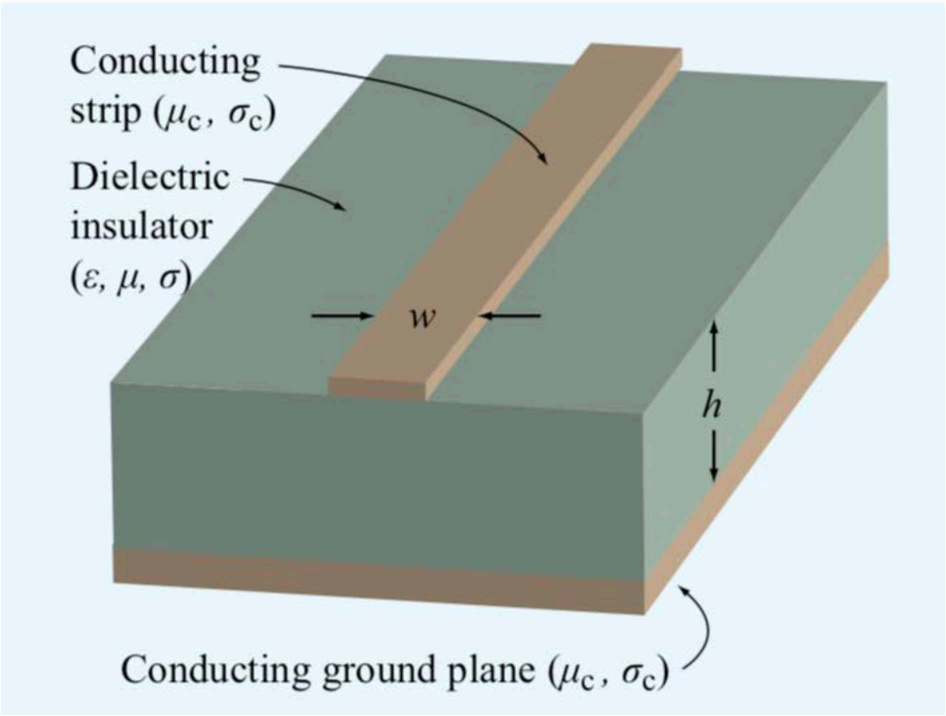
$$\Gamma = e^{-j180^o}\ e^{j119.3^o}(0.76) = 0.76\angle -60.7°$$

$|\Gamma|$ θ_r

Reflection Coefficient $\quad \Gamma = |\Gamma| e^{j\theta_r}$

Load	$\lvert\Gamma\rvert$	θ_r
 Z_0 (Matched Load)	0 (no reflection)	irrelevant
 (short)	1	$\pm 180°$ (phase opposition)
 (open)	1	0 (in-phase)
 $jX = j\omega L$ (Inductive Load)	1	$\pm 180° - 2\tan^{-1} x$
 $jX = \dfrac{-j}{\omega C}$ (Capacitive Load)	1	$\pm 180° + 2\tan^{-1} x$

Microstrip Line



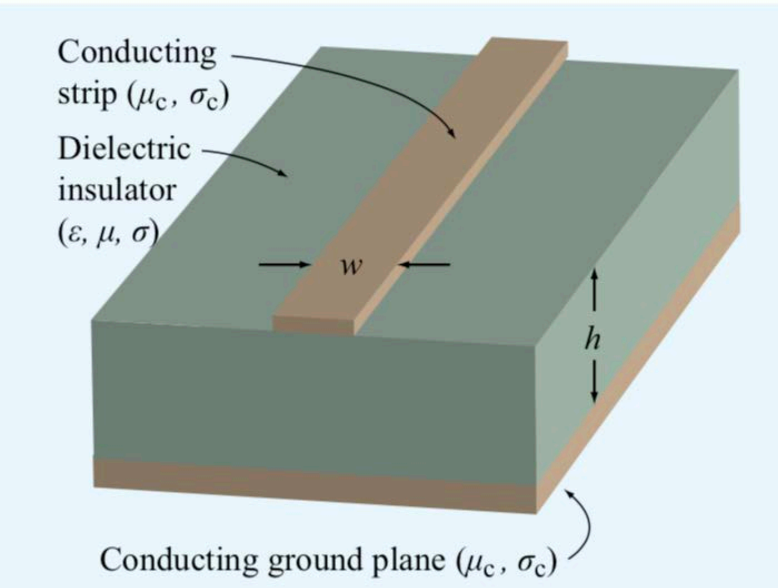
- Used in RF and microwave (high frequency) circuits
- Can fabricate on PCB

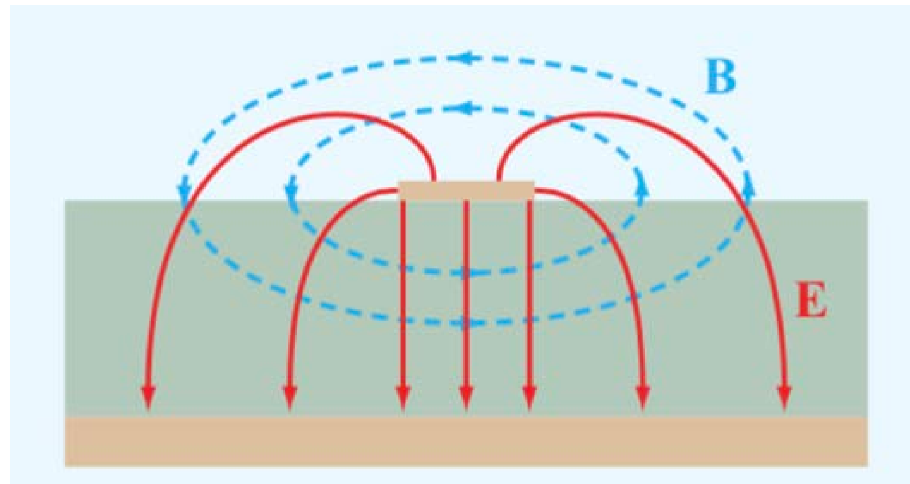
Geometric parameters:

- w = width
- h = thickness of dielectric

- We ignore strip thickness as long as << w

Microstrip Line





Region between conductors have approximately $\vec{E} \perp \vec{B}$
Quasi-TEM → will model as TEM

Substrate = perfect dielectric, $\sigma = 0$
Metal strip / ground conductors = perfect conductors, $\sigma \rightarrow \infty$
Also, $\mu = \mu_0$

Microstrip Line

The key parameters become: w, h, ϵ

If $\vec{E}$ is confined between conductors:

$$u_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{\epsilon_r}} \qquad \beta = \frac{\omega}{c}\sqrt{\epsilon_r}$$

where ϵ_r = relative permittivity of dielectric

Since some $\vec{E}$ is through air, we define: ϵ_{eff} = effective permittivity

$$u_p = \frac{c}{\sqrt{\epsilon_{eff}}}$$

With significant derivation:

$$\epsilon_{eff} = \frac{\epsilon_r + 1}{2} + \left(\frac{\epsilon_r - 1}{2}\right)\left(1 + \frac{10}{s}\right)^{-xy}$$

where s = $\frac{w}{h}$ (width to thickness ratio)

Microstrip Line

From previous slide:

$$\epsilon_{eff} = \frac{\epsilon_r + 1}{2} + \left(\frac{\epsilon_r - 1}{2}\right)\left(1 + \frac{10}{s}\right)^{-xy}$$

x, y → additional variables → used to obtain:

$$Z_0 = \frac{60}{\sqrt{\epsilon_{eff}}} \ln\{\frac{6 + (2\pi - 6)e^{-t}}{s} + \sqrt{1 + \frac{4}{s^2}}\}$$

Characteristic Impedance

$$t = \left(\frac{30.67}{s}\right)^{0.75}$$

Generalized, Lossless Transmission Line

Transmission line can be characterized by:

- Propagation constant: γ
- Characteristic impedance: Z_0

→ Then can specify ω, R', L', G', C'

Want TL with low ohmic loss → get high σ_c conductors and dielectrics with $\sigma \to 0$ then, R', G' can be made small and negligible

If $R' << \omega L'$ and $G' << \omega C'$, then we can set $R' = G' \cong 0$

$$\gamma = \sqrt{(\cancel{R'} + j\omega L')(\cancel{G'} + jwC')}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{L'C'} \longrightarrow \text{Lossless line}$$

$$\alpha = 0, \qquad \beta = \omega\sqrt{L'C'}$$

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{\cancel{R'} + j\omega L'}{\cancel{G'} + jwC'}} = \sqrt{\frac{L'}{C'}} = Z_0$$

Lossless Transmission Line

For lossless transmission line:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{L'C'}} \qquad u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$

Recall: all <u>TEM lines L'C' = με</u>

Then: $\beta = \omega\sqrt{\mu\epsilon}\ \frac{\text{rad}}{\text{m}}$

where μ, ϵ -- dielectric
usually $\mu = \mu_0$
$\epsilon = \epsilon_r \epsilon_0$

$$u_p = \frac{1}{\sqrt{\mu\epsilon}}\ \text{m/s}$$

$$u_p = \frac{1}{\sqrt{\mu_0\epsilon_r\epsilon_0}} = \frac{1}{\sqrt{\mu_0\epsilon_0}}\frac{1}{\sqrt{\epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}}$$

If $\epsilon_r = 1$, then $u_p = c$

$$\lambda = \frac{u_p}{f} = \frac{c}{f}\frac{1}{\sqrt{\epsilon_r}} = \frac{\lambda_0}{\sqrt{\epsilon_r}}$$

Lossless Transmission Line

Define $\epsilon_r = \frac{\epsilon}{\epsilon_0}$, $\quad u_\mathrm{p} = \dfrac{1}{\sqrt{\mu_0 \epsilon_\mathrm{r} \epsilon_0}} = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}} \cdot \dfrac{1}{\sqrt{\epsilon_\mathrm{r}}} = \dfrac{c}{\sqrt{\epsilon_\mathrm{r}}}$,

Guided wavelength, $\lambda = \dfrac{u_p}{f} = \dfrac{c}{f}\dfrac{1}{\sqrt{\epsilon_r}} = \dfrac{\lambda_0}{\sqrt{\epsilon_r}}$ In air (or vacuum) $\lambda_0 = \frac{c}{f}$

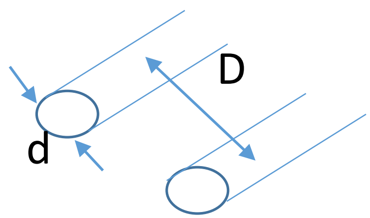
If ϵ_r is independent of frequency → non-dispersive
u_p is independent of frequency, no dispersion
Same phase velocity for all frequency components

→ Short pulse → more frequencies,
dispersion is more critical at high data rates

TL– characteristic parameters -- Summary

	Propagation Constant $\gamma = \alpha + j\beta$	Phase Velocity u_p	Characteristic Impedance Z_0
General Case:	$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$ $\mu = \mu_0, \epsilon = \epsilon_r \epsilon_0,$	$u_p = \dfrac{\omega}{\beta}$ $c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}, \sqrt{\dfrac{\mu_0}{\epsilon_0}} \approx 120\pi\Omega$ (free space)	$Z_0 = \sqrt{\dfrac{(R' + j\omega L')}{(G' + j\omega C')}}$
Lossless (R' = G'= 0)	$\gamma = j\beta \quad (\alpha = 0)$ $\beta = \dfrac{\omega\sqrt{\epsilon_r}}{c}$ Non-dispersive line: ϵ_r of insulator is independent of frequency	$u_p = \dfrac{c}{\sqrt{\epsilon_r}}$	$Z_0 = \sqrt{\dfrac{L'}{C'}}$
Lossless coax	$\alpha = 0$ $\beta = \dfrac{\omega\sqrt{\epsilon_r}}{c}$	$u_p = \dfrac{c}{\sqrt{\epsilon_r}}$	$Z_0 = \dfrac{60}{\sqrt{\epsilon_{eff}}} \ln(\dfrac{b}{a})$ $z_0 = \sqrt{\dfrac{\frac{\mu}{2\pi}\ln(\frac{b}{a})}{\frac{2\pi\epsilon}{\ln(\frac{b}{a})}}} = \sqrt{\dfrac{\mu}{\epsilon}}\dfrac{1}{2\pi}\ln(\dfrac{b}{a})$

TL– characteristic parameters -- Summary

	Propagation Constant $\gamma = \alpha + j\beta$	Phase Velocity u_p	Characteristic Impedance Z_0
Lossless 2-wire:	$\alpha = 0$ $\beta = \frac{\omega\sqrt{\epsilon_r}}{c}$ 	$u_p = \frac{c}{\sqrt{\epsilon_r}}$	$Z_0 = \frac{120}{\sqrt{\epsilon_r}} \ln[\left(\frac{D}{d}\right) + \sqrt{(\frac{D}{d})^2 - 1}]$
Lossless parallel-plate	$\alpha = 0$ $\beta = \frac{\omega\sqrt{\epsilon_r}}{c}$	$u_p = \frac{c}{\sqrt{\epsilon_r}}$	$Z_0 = \frac{120\pi}{\sqrt{\epsilon_r}} (\frac{h}{w})$ separation Plate width